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Hyperscaling Inequalities for the Contact Process and Oriented Percolation

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The contact process and oriented percolation are expected to exhibit the same critical behavior in any dimension. Above their upper critical dimension d_c, they exhibit the same critical behavior as the branching process. Assuming existence of the critical exponents, we prove a pair of hyperscaling inequalities which, together with the results of refs. 16 and 18, implies $d_c = 4$.

KEY WORDS: Contact process; oriented percolation; critical exponent; hyperscaling inequality; upper critical dimension.

1. INTRODUCTION

The contact process (CP) was introduced by Harris[(12)] as a model of spread of an infectious disease. In this paper every individual is supposed to be either healthy or infected and to reside in the d-dimensional integer lattice $\mathbb{Z}^d$, but not to move around in $\mathbb{Z}^d$. It is known that this model exhibits a phase transition and critical behavior (see Section 3).

By the graphical representation (see Section 2), CP in $\mathbb{Z}^d$ is considered as oriented percolation (OP) in $\mathbb{Z}^d \times \mathbb{R}_+$ and is believed to exhibit the same critical behavior as OP in $\mathbb{Z}^d \times \mathbb{Z}_+$.

The main ingredient which makes analyses for CP and OP difficult is interaction between distinct sites; offspring of one site is not independent of that of another site. The branching process is a model where offspring are independent each other, and we can easily solve its problem. We refer to the critical behavior of the branching process as the *mean-field (MF) behavior*.

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In high dimensions, the interaction among distinct sites is expected to be weak. It has been proved[16, 18] that the *nearest-neighbor* CP and OP with $d \gg 4$ and sufficiently *spread-out* CP and OP with $d > 4$, defined in Section 2, satisfy a diagrammatic condition, called the *triangle condition,*[2–4] which implies that both models exhibit the MF behavior. Under the *universality* hypothesis, the critical behavior of the nearest-neighbor CP and OP is believed to be identical to that of the spread-out CP and OP in any dimension, and thus the value of the *upper critical dimension* d_c is expected to be at most 4: any spatially symmetric finite-range CP and OP exhibit the MF behavior for any $d > 4$.

In this paper we derive a pair of *hyperscaling inequalities* for CP and OP (Corollary 4.2) which implies $d_c \geqslant 4$. These inequalities relate the spatial dimension d to the *critical exponents* which indicate singular behavior of observables in the vicinity of the critical point (see Section 3). The inequalities are proved by assuming existence of the critical exponents. The proof is based on argument similar to that in refs. 19 and 20 for unoriented percolation.

We organize the rest of this paper as follows. In Section 2 we define the nearest-neighbor and spread-out CP and OP. In Section 3 we present several results obtained so far about the phase transition and critical behavior of the models defined in Section 2. In Section 4 we present the pair of hyperscaling inequalities. Finally in Section 5 we prove the hyperscaling inequalities by assuming existence of the critical exponents introduced in Section 3.

2. MODELS

We here define the models to be discussed. Let $\Omega \subset \mathbb{Z}^d$ denote a fixed set of sites which is symmetric with respect to the symmetries of the lattice. We suppose that its cardinality $|\Omega|$ is finite. For $x \in \mathbb{Z}^d$, we define $D(x)$ to be $1/|\Omega|$ if $x \in \Omega$, and 0 otherwise.

Oriented Percolation (OP). We think of $\mathbb{Z}^d \times \mathbb{Z}_+$ as space-time. A *bond* is defined to be an ordered pair $[(x, t), (y, t+1)\rangle$ of sites in $\mathbb{Z}^d \times \mathbb{Z}_+$. For any $t \in \mathbb{Z}_+$, a bond $[(x, t), (y, t+1)\rangle$ is ether *open* or *closed* with probability $\lambda D(y-x)$ and $1-\lambda D(y-x)$ respectively, independently of the other bonds, where $\lambda \in [0, |\Omega|]$.

A site (x, s) is said to be *connected to* (y, t) if either $(x, s) = (y, t)$ or there are $t-s$ open bonds $\{[(z_{i-1}, s+i-1), (z_i, s+i)\rangle\}_{i=1}^{t-s}$ satisfying that $z_0 = x$ and $z_{t-s} = y$; we denote $(x, s) \to (y, t)$ for this event. In particular we define $\mathbf{C}_t = \{x \in \mathbb{Z}^d : (o, 0) \to (x, t)\}$.

Contact Process (CP). Let $\mathbf{C}_t \subset \mathbb{Z}^d$ denote the set of infected individuals at time $t \in \mathbb{R}_+$, provided that $\mathbf{C}_0 = \{o\}$. The dynamical process of $\mathbf{C}_t$ is defined as

$$\begin{cases} x \in \mathbf{C}_t \longrightarrow x \notin \mathbf{C}_{t+\varepsilon}, & \text{with probab. } \varepsilon + o(\varepsilon), \\ x \notin \mathbf{C}_t \longrightarrow x \in \mathbf{C}_{t+\varepsilon}, & \text{with probab. } \lambda\varepsilon \sum_{y \in \mathbf{C}_t} D(x-y) + o(\varepsilon), \end{cases}$$

where $\lambda \geqslant 0$ and $o(\varepsilon)$ denotes a function satisfying $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \downarrow 0$.

This model can also be constructed by the *graphical representation* (ref. 15 and references therein) as follows: we consider $\mathbb{Z}^d \times \mathbb{R}_+$ as space-time. Along each time line $\{x\} \times \mathbb{R}_+$, we place points in the manner of a Poisson process with intensity 1, independently of the other points. And for each ordered pair of distinct time lines from $\{x\} \times \mathbb{R}_+$ to $\{y\} \times \mathbb{R}_+$, we place bonds $\{[(x, t), (y, t)\rangle\}_{t \geqslant 0}$ in the manner of a Poisson process with intensity $\lambda D(y-x)$, independently of the other Poisson processes.

A site (x, s) is said to be *connected to* (y, t) if either $(x, s) = (y, t)$ or there is a path in $\mathbb{Z}^d \times \mathbb{R}_+$ from (x, s) to (y, t) using the equipped bonds and time line segments traversed in the increasing time direction without traversing the equipped points; we denote $(x, s) \to (y, t)$ for this event. The law of $\mathbf{C}_t$ defined above is equivalent to that of $\{x \in \mathbb{Z}^d : (o, 0) \to (x, t)\}$.

We refer to the model defined by $\Omega = \{x \in \mathbb{Z}^d : \sum_i |x_i| = 1\}$ as the *nearest-neighbor model*, and to the model defined by $\Omega = \{x \in \mathbb{Z}^d : 0 < \max_i |x_i| \leqslant L\}$ for some $L < \infty$ as the *spread-out model.*[2] We write $\mathbb{P}_\lambda$ for the associated probability measure of both models. Although we exclusively discuss CP, the rest of this paper is also applicable to OP.

3. PHASE TRANSITION AND CRITICAL BEHAVIOR

We here present several results obtained so far about the phase transition and critical behavior of CP and OP. First we define several quantities to be discussed. The *order parameter* $\theta(\lambda)$ is defined to be $\mathbb{P}_\lambda(\mathbf{C}_\infty \neq \varnothing)$, which is a decreasing limit of $\theta_t(\lambda) \equiv \mathbb{P}_\lambda(\mathbf{C}_t \neq \varnothing)$ as $t \uparrow \infty$. We also define $\Theta_t(\lambda) = \theta_t(\lambda) - \theta(\lambda)$ and its *relaxation time*

$$\tau(\lambda) = \inf\{T \geqslant 0 : \Theta_t(\lambda) \leqslant e^{-t/T}, \forall t \geqslant 0\}. \tag{3.1}$$

And we define the *connectivity function* $\phi_\lambda(x, t) = \mathbb{P}_\lambda((o, 0) \to (x, t))$ and the *susceptibility* $\chi(\lambda) = \int_0^\infty dt\, \chi_t(\lambda)$ (which is defined to be $\sum_{t \in \mathbb{Z}_+} \chi_t(\lambda)$

[2] We can consider a more general class as the spread-out model. See refs. 4, 10, and 13 for example.

for OP) with $\chi_t(\lambda) = \sum_x \phi_\lambda(x, t)$. Finally we define the *radius of gyration* at time t

$$\xi_t(\lambda) = \left\{ \frac{1}{\chi_t(\lambda)} \sum_x |x|^2 \phi_\lambda(x, t) \right\}^{1/2}, \qquad \text{where} \quad |x|^2 = \sum_{i=1}^d x_i^2.$$

For the nearest-neighbor CP and OP with $d \geqslant 1$ and for the spread-out OP with $d \geqslant 1$ and $L \geqslant 1$, it has been proved[5, 10] that there exists a *critical point* $\lambda_c \in (0, \infty)$ ($\lambda_c \in (0, |\Omega|)$ for OP) such that $\theta(\lambda) = 0$ when and only when $\lambda \leqslant \lambda_c$. Therefore $\Theta_t(\lambda) = \theta_t(\lambda)$ when $\lambda \leqslant \lambda_c$. We note that, by the super-multiplicative property $\theta_{s+t}(\lambda) \geqslant \theta_s(\lambda)\, \theta_t(\lambda)$, the relaxation time for $\lambda \leqslant \lambda_c$ satisfies

$$e^{-1/\tau(\lambda)} = \lim_{t \uparrow \infty} \theta_t(\lambda)^{1/t} = \sup_{t > 0} \theta_t(\lambda)^{1/t}. \tag{3.2}$$

It is known (ref. 15, p. 57) that $\Theta_t(\lambda)$ also decays exponentially as $t \uparrow \infty$ when $\lambda > \lambda_c$.

For the nearest-neighbor CP and OP with $d \geqslant 1$ and for the spread-out CP and OP with $d \geqslant 1$ and $L \geqslant 1$, it has been proved[1, 2, 4, 6] that $\chi(\lambda)$ is finite when $\lambda < \lambda_c$ and diverges as $\lambda \uparrow \lambda_c$. In fact

$$\chi(\lambda) \geqslant (\lambda_c - \lambda)^{-1}, \qquad \forall \lambda < \lambda_c. \tag{3.3}$$

It is generally believed that there exists a *critical exponent* γ such that

$$\chi(\lambda) \approx (\lambda_c - \lambda)^{-\gamma}, \qquad \text{as} \quad \lambda \uparrow \lambda_c. \tag{3.4}$$

This means that there exist $s_1(\varepsilon), s_2(\varepsilon) \in (0, \infty)$ slowly varying[3] as $\varepsilon \downarrow 0$ such that for $\varepsilon = \lambda_c - \lambda$

$$s_1(\varepsilon) \leqslant \frac{\chi(\lambda)}{(\lambda_c - \lambda)^{-\gamma}} \leqslant s_2(\varepsilon), \tag{3.5}$$

holds for any positive ε close to 0. The inequality (3.3) implies $\gamma \geqslant 1$ if the exponent γ exists. Under the *triangle condition*,[2, 4] it has been proved that $\chi(\lambda)$ is bounded from above by a λ-independent multiple of $(\lambda_c - \lambda)^{-1}$ for any $\lambda < \lambda_c$ close to λ_c, and thus γ exists and equals 1 for the nearest-neighbor CP and OP and for the spread-out CP and OP with $L \geqslant 1$. The exponent γ as well as the other critical exponents defined below is believed to be *universal*: γ depends only on the value of d and spatial symmetry of the

[3] A function $s(\varepsilon)$ is said to be *slowly varying* as $\varepsilon \downarrow 0$ if $\lim_{\varepsilon \downarrow 0} s(c\,\varepsilon)/s(\varepsilon) = 1$ for every $c > 0$. An example of $s(\varepsilon)$ is $|\ln \varepsilon|$.

models, but not on whether the model is nearest-neighbor or spread-out. The dimension-independent value $\gamma = 1$ is called the *MF value* of γ, which is the critical exponent for the branching process: for the (continuous-time) branching process, $\chi_t(\lambda)$ is the expected number of sites in $\mathbb{Z}^d$ where at least one offspring exists, and satisfies the differential equation

$$\frac{\partial}{\partial t}\chi_t(\lambda) = (\lambda - 1)\,\chi_t(\lambda), \qquad \text{with} \quad \chi_0(\lambda) = 1.$$

We can easily solve this to obtain $\chi(\lambda) = (1-\lambda)^{-1}$ for $\lambda < 1$.

It has been proved[16, 18] that the triangle condition holds and thus $\gamma = 1$ for the nearest-neighbor CP and OP with $d \gg 4$ and for the spread-out CP and OP with $d > 4$ and $L \gg 1$. The universality hypothesis stands behind the belief that any spatially symmetric finite-range CP and OP exhibit the MF behavior for any $d > 4$.

The other critical exponents discussed in this paper are defined as follows.

$$\tau(\lambda) \approx \begin{cases} (\lambda_c - \lambda)^{-\zeta}, & \text{as} \quad \lambda \uparrow \lambda_c, \\ (\lambda - \lambda_c)^{-\zeta'}, & \text{as} \quad \lambda \downarrow \lambda_c, \end{cases} \tag{3.6}$$

$$\theta(\lambda) \approx (\lambda - \lambda_c)^{\beta}, \qquad \text{as} \quad \lambda \downarrow \lambda_c, \tag{3.7}$$

$$\theta_t(\lambda_c) \approx t^{-\rho}, \qquad \text{as} \quad t \uparrow \infty, \tag{3.8}$$

$$\xi_t(\lambda_c) \approx t^{\nu}, \qquad \text{as} \quad t \uparrow \infty, \tag{3.9}$$

where the behavior as $t \uparrow \infty$ is defined in the same way as (3.5) with $\varepsilon = 1/t$. It is believed that $\zeta = \zeta'$ holds in any dimension. The MF values of ζ, ζ', β and ρ are all equal to 1, which are obtained by solving

$$\frac{\partial}{\partial t}\theta_t(\lambda) = (\lambda - 1)\,\theta_t(\lambda) - \lambda\theta_t(\lambda)^2, \qquad \text{with} \quad \theta_0(\lambda) = 1.$$

For the nearest-neighbor CP and OP and for the spread-out CP and OP with $L \geqslant 1$, it has been proved that $\beta \leqslant 1$ if the exponent β exists,[1, 4, 6] and that β exists and equals 1 under the triangle condition.[3, 4] It has also been proved[13, 17] that the exponent ν exists and equals $1/2$, and the Gaussian scaling limit

$$\lim_{t \uparrow \infty} \frac{1}{\chi_t(\lambda_c)} \sum_x \phi_{\lambda_c}(x, t)\, e^{ik \cdot x/\sqrt{t}} = e^{-A|k|^2}, \qquad \text{for some} \quad A > 0,$$

holds for the nearest-neighbor OP with $d \gg 4$ and for the spread-out OP with $d > 4$ and $L \gg 1$. These results are associated with *super-Brownian motion* (ref. 14 and references therein).

4. MAIN RESULTS

Except for the above cases, there is still no proof of existence of the critical exponents. Assuming their existence and establishing the following theorem, we in this paper prove the *hyperscaling inequalities* (4.3) and (4.4).

Theorem 4.1.

1. Let $T = q\tau(\lambda)\ln\tau(\lambda)$ with $q \in (d+1, \infty)$ and $t = \tau(\lambda)^r$ with $r \in (0, \frac{1}{d+1})$. Then there exists $\lambda_0 < \lambda_c$ such that

$$\chi(\lambda) \leqslant 4 \int_t^T ds \{4\xi_s(\lambda_c)+1\}^d \, \theta_{s/2}(\lambda_c)^2, \qquad \forall \lambda \in (\lambda_0, \lambda_c). \tag{4.1}$$

2. For every $\lambda > \lambda_c$,

$$\theta_t(\lambda_c) \leqslant 2\,\theta(\lambda), \qquad \forall t \geqslant \tau(\lambda) \ln \frac{1}{\theta(\lambda)}. \tag{4.2}$$

Corollary 4.2. Assuming existence of the critical exponents in (3.4) and (3.6)–(3.9), we have

$$(d\nu - 2\rho + 1)\,\zeta \geqslant \gamma, \tag{4.3}$$

$$(d\nu + 1)\,\bar{\zeta} \geqslant \gamma + 2\beta, \tag{4.4}$$

where $\bar{\zeta} = \zeta \vee \zeta'$.

We obtain $d \geqslant 4$ by substituting the MF values to the above inequalities. This means that the MF exponents cannot be observed when $d < 4$. Together with the results of refs. 16 and 18, this implies that the value of the *upper critical dimension* d_c is 4.

The proof of the above theorem is based on argument similar to that in refs. 19 and 20[4] where a set of hyperscaling inequalities for d-dimensional unoriented percolation has been proved by assuming existence of its critical exponents; other hyperscaling inequalities have also been derived in

[4] Tasaki[19] discussed a class of random cluster models which includes not only unoriented percolation but also some stochastic cluster growth models. We note that CP and OP are different from the cluster growth models; the susceptibility for the growth models, which corresponds to $\lim_{t\uparrow\infty} \chi_t(\lambda)$ in this paper, diverges in any dimension as the control parameter tends to its critical point. In contrast, it has been proved[13, 17] that $\chi_t(\lambda_c)$ is bounded uniformly in t for the nearest-neighbor OP with $d \gg 4$ and for the spread-out OP with $d > 4$ and $L \gg 1$.

refs. 7 and 8. Their results, together with the results of ref. 11, imply $d_c = 6$ for unoriented percolation. Borgs *et al.*[7] have also proved a sufficient condition for a set of hyperscaling *equalities* (see Remark in the next section).

5. PROOFS

We here prove Theorem 4.1 and then Corollary 4.2. First we prove the inequality (4.1) by assuming the following three lemmas, whose proofs are given after proving Corollary 4.2.

Lemma 5.1. Let $T = q\tau(\lambda) \ln \tau(\lambda)$ with $q \in (d+1, \infty)$ and $t = \tau(\lambda)^r$ with $r \in (0, \frac{1}{d+1})$. Then there exists $\lambda_0 < \lambda_c$ such that

$$\chi(\lambda) \leqslant 3I_\lambda(t, T), \qquad \forall \lambda \in (\lambda_0, \lambda_c), \tag{5.1}$$

where $I_\lambda(t, T) = \int_t^T ds\, \chi_s(\lambda)$.

Lemma 5.2.

$$\phi_\lambda(x, t) \leqslant \theta_{t/2}(\lambda)^2, \qquad \forall \lambda \geqslant 0, \quad \forall (x, t) \in \mathbb{Z}^d \times \mathbb{R}_+. \tag{5.2}$$

Lemma 5.3. For any $\lambda \leqslant \lambda_c$, there exists $C < \infty$ such that

$$e^{-t/\tau(\lambda)} \leqslant \chi_t(\lambda) \leqslant C\,(t+1)^d\, e^{-t/\tau(\lambda)}, \qquad \forall t \geqslant 0. \tag{5.3}$$

The reversed inequality of (5.1), $\chi(\lambda) \geqslant I_\lambda(t, T)$, is trivial. Therefore Lemma 5.1 implies that $I_\lambda(t, T)$ approximates $\chi(\lambda)$ by taking λ to be close to λ_c according to the integral region $[t, T]$. The inequality (5.3) is used to prove Lemma 5.1. It also indicates that $\tau(\lambda)$ diverges as $\lambda \uparrow \lambda_c$ together with the fact that $\chi(\lambda)$ diverges as $\lambda \uparrow \lambda_c$ as in (3.3).

Proof of Theorem 4.1. First we prove (4.1). By the definition of $\xi_t(\lambda)$,

$$\xi_t(\lambda)^2 \geqslant \frac{1}{\chi_t(\lambda)} \sum_{x:\, |x| \geqslant R} |x|^2\, \phi_\lambda(x, t) \geqslant \frac{R^2}{\chi_t(\lambda)} \sum_{x:\, |x| \geqslant R} \phi_\lambda(x, t),$$

holds for any $R \geqslant 0$. We substitute $R = 2\,\xi_t(\lambda)$ to obtain

$$\frac{3}{4}\,\chi_t(\lambda) \leqslant \sum_{x:\, |x| \leqslant 2\xi_t(\lambda)} \phi_\lambda(x, t). \tag{5.4}$$

Applying Lemma 5.1, the monotonicity of $\chi_t(\lambda)$ in λ, the above inequality (5.4), and then Lemma 5.2, we obtain the inequality (4.1).

Next we prove (4.2). By the definition (3.1),

$$\theta_t(\lambda) = \theta(\lambda) + \Theta_t(\lambda) \leqslant \theta(\lambda) + e^{-t/\tau(\lambda)}, \qquad \forall t \geqslant 0. \tag{5.5}$$

Let $\lambda > \lambda_c$ so that $\theta(\lambda) > 0$ and $\tau(\lambda) < \infty$. The right side of (5.5) is bounded by $2\,\theta(\lambda)$ when $t \geqslant \tau(\lambda) \ln \frac{1}{\theta(\lambda)}$. Together with the monotonicity of $\theta_t(\lambda)$ in λ, we obtain the inequality (4.2). This completes the proof of Theorem 4.1. ▌

Proof of Corollary 4.2. First we derive the hyperscaling inequality (4.3) by using the inequality (4.1) and assuming existence of the exponents γ, ζ, ρ and ν.

We take λ to be close to λ_c so that t in (4.1) is large and thus the integrand in (4.1) is bounded by $s^{d\nu - 2\rho}$ multiplied by some slowly varying function. We note that the exponent $d\nu - 2\rho$ must be greater than -1 because of the fact that $\chi(\lambda)$ diverges as $\lambda \uparrow \lambda_c$ as in (3.3). Therefore the right side of (4.1) is bounded by $\tau(\lambda)^{d\nu - 2\rho + 1}$ multiplied by some slowly varying function. Together with the assumed behavior of $\chi(\lambda)$ in (3.4) and that of $\tau(\lambda)$ in (3.6), we obtain the inequality (4.3).

Although the inequality (4.3) involves only the critical exponents for $\lambda \leqslant \lambda_c$, the other hyperscaling inequality (4.4) involves ζ' and β as well. The inequality (4.4) follows from the scaling inequality[5] $\rho\zeta' \geqslant \beta$ by substituting it to (4.3). We prove this scaling inequality by using the inequality (4.2) and assuming existence of the exponents ζ', β and ρ.

Let $\lambda_t = \inf\{\lambda > \lambda_c : \tau(\lambda) \ln \frac{1}{\theta(\lambda)} \leqslant t\}$. Then the inequality (4.2) holds at $\lambda = \lambda_t$, and $(\lambda_t - \lambda_c)^{-\zeta'} \approx t$, which follows from $\tau(\lambda_t) \ln \frac{1}{\theta(\lambda_t)} = t$, holds under the assumed behavior of $\tau(\lambda)$ in (3.6) and that of $\theta(\lambda)$ in (3.7). Taking t to be large so that

$$\theta_t(\lambda_c) \approx t^{-\rho} \approx (\lambda_t - \lambda_c)^{\rho\zeta'}, \qquad \theta(\lambda_t) \approx (\lambda_t - \lambda_c)^{\beta},$$

and substituting them to the inequality (4.2), we obtain $\rho\zeta' \geqslant \beta$. This completes the proof of Corollary 4.2. ▌

Remark. Because of the inequality (5.2), we cannot derive the corresponding equality to (4.3). For unoriented percolation, an analogous inequality to (5.2) holds and thus its hyperscaling inequality can be derived.[7,20] Borgs *et al.*[7] have proved for d-dimensional unoriented percolation

[5] When $d = 1$, it has been proved (ref. 9, p. 73) that $t^{1/2}\theta_t(\lambda_c)$ diverges as $t \uparrow \infty$. This means $\rho \leqslant 1/2$ for $d = 1$ if the exponent ρ exists. The numerical values of ρ and β for $d = 1$, 0.161 and 0.277 respectively, have also been mentioned in ref. 9.

that an inequality, which is a sort of reversed version of (5.2) and is expected to hold only when $d < d_c$, holds and thus the hyperscaling equality holds if the expected number of disjoint easy-way crossings in the box $[0, L] \times [0, 3L]^{d-1}$ is bounded uniformly in L. We expect that a similar situation occurs in CP and OP as well, which would help explain why the inequality (5.2) can be reversed in low dimensions.

Proof of Lemma 5.3. First we prove that for any $\lambda \geqslant 0$ there exists $C < \infty$ such that

$$\theta_t(\lambda) \leqslant \chi_t(\lambda) \leqslant C(t+1)^d\, \theta_t(\lambda), \qquad \forall t \geqslant 0, \tag{5.6}$$

which is trivial for OP. We define $P_t^\lambda(n)$ to be the probability that the number of sites in $\mathbf{C}_t$ equals n. It has been proved (ref. 15, p. 42) that for any $\lambda \geqslant 0$ there exists $C < \infty$ such that

$$\sum_{n \geqslant 1} n^k\, P_t^\lambda(n) \leqslant C^k (t+1)^{kd}, \qquad \forall k \geqslant 1, \quad \forall t \geqslant 0.$$

Together with the Hölder inequality, we have for any $k \geqslant 1$

$$\begin{aligned} \chi_t(\lambda) = \sum_{n \geqslant 1} n\, P_t^\lambda(n) &\leqslant \left\{ \sum_{n \geqslant 1} n^k P_t^\lambda(n) \right\}^{1/k} \left\{ \sum_{n \geqslant 1} P_t^\lambda(n) \right\}^{1-1/k} \\ &\leqslant C(t+1)^d\, \theta_t(\lambda)^{1-1/k}. \end{aligned}$$

The inequality (5.6) is obtained by taking $k \uparrow \infty$.

By the sub-multiplicative property $\chi_{s+t}(\lambda) \leqslant \chi_s(\lambda)\, \chi_t(\lambda)$, there exists $\sigma(\lambda) \geqslant 1$ such that

$$e^{-1/\sigma(\lambda)} = \lim_{t \uparrow \infty} \chi_t(\lambda)^{1/t} = \inf_{t > 0} \chi_t(\lambda)^{1/t}. \tag{5.7}$$

The inequality (5.6) indicates that $\sigma(\lambda)$ is equivalent to the relaxation time $\tau(\lambda)$ when $\lambda \leqslant \lambda_c$. We thus obtain (5.3) by using (3.2), (5.6) and (5.7). The proof is completed. ∎

Proof of Lemma 5.1. By the inequality (5.3),

$$I_\lambda(0, t) \leqslant C(t+1)^{d+1}, \qquad I_\lambda(T, \infty) \leqslant Ce \int_{T+1}^{\infty} ds\, s^d e^{-s/\tau(\lambda)},$$

$$I_\lambda(t, T) \geqslant \tau(\lambda)\{e^{-t/\tau(\lambda)} - e^{-T/\tau(\lambda)}\}.$$

Since $t = \tau(\lambda)^r$ with $r \in (0, \frac{1}{d+1})$, the inequality $I_\lambda(0, t) \leqslant I_\lambda(t, T)$ holds when $\tau(\lambda)$ is sufficiently large.

To estimate $I_\lambda(T,\infty)$, we define $f(s)$ as $e^{-f(s)\,s/\tau(\lambda)}=s^d e^{-s/\tau(\lambda)}$; we note that $f(s)$ is increasing in $s>e$. We substitute $T=q\tau(\lambda)\ln\tau(\lambda)$ in $f(T)$ to obtain

$$f(T)=\frac{q-d}{q}-\frac{d}{q}\,\frac{\ln\ln\tau(\lambda)+\ln q}{\ln\tau(\lambda)}.$$

Since $q>d+1$, $f(T)$ is positive when $\tau(\lambda)$ is sufficiently large. Now we can bound $I_\lambda(T,\infty)$ by

$$Ce\int_T^\infty ds\, e^{-sf(T)/\tau(\lambda)}=\frac{Ce\tau(\lambda)^{1-qf(T)}}{f(T)}.$$

Again by the condition $q>d+1$, $1-qf(T)$ is negative when $\tau(\lambda)$ is sufficiently large and thus $I_\lambda(T,\infty)$ is small. This completes the proof of Lemma 5.1. ∎

Proof of Lemma 5.2. By the Markov property,

$$\begin{aligned}\mathbb{P}_\lambda((o,0)\to(x,t))&=\mathbb{P}_\lambda(\exists y\in\mathbb{Z}^d,(o,0)\to(y,t/2)\to(x,t))\\&\leqslant\mathbb{P}_\lambda(\exists y,z\in\mathbb{Z}^d,(o,0)\to(y,t/2),(z,t/2)\to(x,t))\\&=\theta_{t/2}(\lambda)\,\mathbb{P}_\lambda(\exists z\in\mathbb{Z}^d,(z,t/2)\to(x,t)).\end{aligned}$$

Reversing the directions of the temporal axis and of the bonds used in the graphical representation, and then using the symmetry of the Poisson process, we obtain

$$\mathbb{P}_\lambda(\exists z\in\mathbb{Z}^d,(z,t/2)\to(x,t))=\theta_{t/2}(\lambda).$$

This completes the proof of Lemma 5.2. ∎

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